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On New Classes of Solutions of Nonlinear Partial Differential Equations in the Form of Convergent Special Series

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Abstract. The method of special series with recursively calculated coefficients is used to solve nonlinear partial differential equations. The recurrence of finding the coefficients of the series is achieved due to a special choice of functions, in powers of which the solution is expanded in a series. We obtain a sequence of linear partial differential equations to find the coefficients of the series constructed. In many cases, one can deal with a sequence of linear ordinary differential equations. We construct classes of solutions in the form of convergent series for a certain class of nonlinear evolution equations. A new class of solutions of generalized Boussinesque equation with an arbitrary function in the form of a convergent series is constructed.

INTRODUCTION

Construction of solutions of differential equations in the form of series with recurrently calculated coefficients is a common approach to studying of the properties of such equations. In the works [1, 2, 3, 4, 5] a method is proposed for constructing a solution of nonlinear partial differential equations in the form of series by powers of specially chosen functions with recurrently calculated coefficients.

The construction of direct Picard iterations for ordinary differential equations leads to the construction of solutions in the form of power series. An investigation of the structure of the resulting successive approximations allowed O. Cauchy to prove his theorem for analytic equations. A direct application of these considerations to partial differential equations is impossible.

The method of special series with recurrently calculated coefficients, developed to represent solutions of nonlinear partial differential equations, is closest to the idea of representing solutions of nonlinear ordinary differential equations by series by powers of functions that are determined in turn from other equations. For example, such approaches were considered by A. Lyapunov [6]. In this case, we also obtain series with recurrently calculated coefficients.

An example of analogous series is found in the monograph of N.P. Yerugin [7], where for equation

$$\dot{y} = -y^2 + \sum_{k=3}^{\infty} a_k y^k, \quad a_k = \text{const}, \quad y(0) = y_0 > 0$$

the solution of the "truncated" equation

$$\dot{z} = -z^2, \quad z(0) = z_0 > 0$$

is considered. In this case, for sufficiently small y_0, z_0, the solution of the original equation is representable in the form of a convergent series

$$y = z\left[1 + z\left(c + \sum_{k=1}^{\infty} \alpha_k(c) z^k\right)\right], \quad c = \text{const}$$

with recurrently calculated coefficients $\alpha_k(c)$. In this case, z is a special function that makes it possible to find the coefficients of the series recurrently.

The method of special series, in contrast to the methods of Galerkin type [8], makes it possible to find a solution with a controlled accuracy, since the approaches lead to a chain of finite systems of ordinary differential equations that

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turn out to be linear even for nonlinear solvable equations, which allows us to obtain new results: in some cases, it is possible to prove global convergence constructed of series in unbounded domains of the [1, 9], where the application of numerical methods has principial difficulties. Special series are also used to obtain solutions in bounded domains for solving initial boundary value problems [10, 11, 12, 13]. For Lin-Reissner-Tsien equation describing transonic flows of gas, various classes of solutions were constructed with the help of special series and problems [14, 15, 16, 17] were solved. One of the stages in the development of the method of special series was the construction of series, where as the zero term of the series it was proposed to use a known exact solution of the investigated equations [11, 18]. In some cases, special series are terminated and exact solutions are obtained [4, 19], which can be used not only for testing the numerical methods [20], but also for choosing the numerical method for solving complex practical problems when investigating the asymptotic of the exact solution [21, 22].

In this paper, we continue development of the approach associated with representation of solutions of nonlinear partial differential equations in the form of special series with recurrently computed coefficients. We consider the general approach to the construction of special series with with a functional arbitrariness and prove the convergence of series of this type for generalized Boussinesque equation.

METHOD OF SPECIAL SERIES. GENERAL APPROACH

Let's consider the partial differential equation with unknown function $u(\mathbf{x}, t)$

$$G\left(u, \ldots, \frac{\partial^{k_0+\ldots+k_m} u}{\partial t^{k_0} \partial z_1^{k_1} \ldots \partial z_m^{k_m}}, \ldots\right) = 0, \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_m)$, $k_0 + \ldots + k_m \leq N$, N is the order of the equation (1) and G is a function analytical in zero by all variables.

We define a ring $K_{\mathbf{g}}$, with the elements are absolutely convergent series in a certain domain

$$u(\mathbf{x}, t) = \sum_{n=0}^{\infty} \alpha_n(\mathbf{g}) P^n(\mathbf{y}), \tag{2}$$

where $g = t$, or $g = x_l$, $(1 \leq l \leq m)$, or $\mathbf{g} = (x_{i_1}, \ldots, x_{i_s})$, or $\mathbf{g} = (x_{i_1}, \ldots, x_{i_s}, t)$, $(1 \leq s \leq m)$. The vector $\mathbf{y}$ has an analogous structure. For the components of vectors $\mathbf{g}$ and $\mathbf{y}$ the relation $\mathbf{g} \otimes \mathbf{y} = (\mathbf{x}, t)$ is valid.

We shall assume, that functions $\alpha_n(\mathbf{g})$, $P(\mathbf{y})$ have corresponding continuous partial derivatives of N–th order, and the function $P(\mathbf{y})$ satisfies relations:

$$\begin{aligned} \tfrac{\partial P}{\partial t} &= \sum_{j=1}^{\infty} h_{0j}(\mathbf{g}) P^j(\mathbf{y}), \\ \tfrac{\partial P}{\partial x_i} &= \sum_{j=1}^{\infty} h_{ij}(\mathbf{g}) P^j(\mathbf{y}), \quad i = \overline{1, m}. \end{aligned} \tag{3}$$

The functions $h_{0j}(\mathbf{g})$, $h_{ij}(\mathbf{g})$ are sufficiently smooth functions such that the series in the right-hand sides of the equations (3) converge absolutely in some area. The conditions (3) guarantee the invariance of the ring $K_{\mathbf{g}}$ with respect to the operation of partial differentiation. The following assertion is valid

Assertion 1. *If functions $h_{0j}(\mathbf{g})$, $h_{ij}(\mathbf{g})$ provide compatibility of overdeterminated system* (3)*, then series* (2) *is a formal solution of the equation* (1) *when coefficients of series* (2) *are correspondingly recurrently constructed.*

This assertion is checked by substitution series (2) into equation (1) and equating of expressions at the same degrees of $P(\mathbf{y})$. The coefficients of series (2) are already recurrently found as solutions of sequence of linear partial differential equations, or solutions of linear ordinary differential equations for rings K_t and K_{x_l}.

Definition. If we find a solution of an initial problem for nonlinear equation in the form of series by the powers of a special function in the form of series by the powers of special function, which allows the series coefficients to be calculated recurrently, then this function we shall call a basic function (BF).

Example of BF. Consider the following function:

$$P(x,t)=\frac{1}{e^{\beta x+\varphi(t)}+f(t)},\quad f(t)\in C^1[0,+\infty),\quad \varphi(t)=\varphi_0+qt,\quad \beta,\varphi_0,q=\text{const.} \tag{4}$$

The system of differential relations for function (4) has the form:

$$\frac{\partial P}{\partial x}=-\beta P+\beta f P^2,\qquad \frac{\partial P}{\partial t}=-qP+(qf-f')P^2. \tag{5}$$

Relations (5) are a special case of differential relations (3). Consequently, the function (4) is a basic function. Therefore, we can use the series (2), (4) to represent solutions of the equation (1). Consider series K_t (2) in following form:

$$u(x,t)=\sum_{i=0}^{\infty}u_i(t)P^i(x,t)=\sum_{i=0}^{\infty}u_i(t)\frac{1}{(e^{\beta x+\varphi(t)}+f(t))^i}. \tag{6}$$

We will use the series (6) for presentation of solutions of the generalized Boussinesque equation for (x,t) variables.

CONVERGENCE OF SPECIAL SERIES FOR GENERALIZED BOUSSINESQUE EQUATION

Consider generalized Boussinesque equation

$$u_t=\alpha(u^2)_{xx}+u_{xxt},\qquad \alpha>0, \tag{7}$$

with initial date

$$u(x,0)=u_0(x),\qquad x\geq 0. \tag{8}$$

Equation (7) describes the motion of a free surface $u(x,t)$, filtered in a porous fluid medium, taking into account the variation of the horizontal components of the vertical filtration rate.

The first boundary value problem for the equation (7) was considered in the following papers [23, 24, 25]. The following initial and boundary conditions were given for the equation:

$$\begin{gathered} u(x,0)=u_0(x),\qquad 0\leq x\leq L;\\ u(x,0)=h(t),\qquad u(x,L)=H(t),\qquad 0\leq t\leq T. \end{gathered} \tag{9}$$

Then under certain restrictions on the functions $u_0(x)$, $h(t)$, $H(t)$ the existence and uniqueness theorem for the classical solution of (7), (9) was proved.

Let initial condition has the following form of convergent series:

$$u(x,t)=\sum_{i=0}^{\infty}u_{i0}P^i(x,0)=\sum_{i=0}^{\infty}u_{i0}\frac{1}{[e^{\beta x+\varphi_0}+f(0)]^i},\quad u_{ij}=\text{const.} \tag{10}$$

Now, after substitution of series (6), (4) into equation (7) and carrying out conversion taking into account relations (5), for obtaining coefficients of series (6) the sequence of a linear ordinary differential equations will be formed

$$\begin{gathered} u_0\equiv u_{00},\\ u_i'+b_i=R_i(t),\\ u_i(0)=u_{i0},\quad i\geq 1, \end{gathered} \tag{11}$$

where

$$b_i=\frac{2\alpha u_{00}}{1-\frac{1}{\beta^2 i^2}}-iq,\quad R_i(t)=\frac{-\beta^2}{\beta^2 i^2-1}\Big\{u_{i-1}(i-1)\Big[\frac{qf-f'}{\beta^2}+f'(2i-1)-qfi(2i-1)$$

$$-(qf-f')\Big]+u'_{i-1}(i-1)(2i-1)f+u_{i-2}(i-2)(i-1)f[fqi-2f'+(qf-f')(2i-3)]$$
$$+u'_{i-2}(i-2)(i-1)f^2-u_{i-3}(i-3)(i-2)(i-1)(qf-f')f^2$$
$$-2\alpha\sum_{m+n=i}[(mn+n^2)u_mu_n-u_mu_{n-1}f(n-1)(2m+2n-1)$$
$$+(m-1)(n-1)f^2u_{m-1}u_{n-1}+(n-2)(n-1)f^2u_mu_{n-2}]\Big\}.$$

Theorem. *Let following conditions are fulfilled:*
1) $0 \le u_{00} \le M, \quad |u_{i0}| \le \dfrac{M}{5}, \quad M > 0;$
2) *for an arbitrary function $f(t) \in C^1[0,\infty)$ the conditions*
$0 \le f(t) \le d$, $(d=0.02)$, $|f'(t)| \le d_1$, $d_1>0$, $0 \le t \le T$ are valid;
3) *for constants from BF* (4) *the inequalities $q \le -4(5d_1+6\alpha M)$, $\beta \ge 2$, $\varphi_0 > T|q|$ are valid.*
Then the solution of problem (8), (10) *is represented in the form of series* (6) *by the powers of BF* (4) *in domain $G_T = \{(x,t) : x \ge 0, \quad 0 \le t \le T\}$.*

Proof. The proof of this theorem is carried out by mathematical induction. The following inequalities are valid:

$$|u_i(t)| \le M, \quad i \ge 0, \quad t \ge 0, \tag{12}$$
$$|u'_i(t)| \le M_1 i, \quad M_1 = 3M|q|, \quad i \ge 0, \quad t \ge 0. \tag{13}$$

Obviously, the inequalities (12), (13) are valid for $i = 0$, $i = 1$. We use the induction hypothesis and the conditions of the theorem to prove these inequalities. We estimate the right part R_{N+1}.

$$|R_{N+1}| \le \frac{4M}{3(N+1)^2}\Big\{\frac{N|q|d}{\beta^2}+\frac{Nd_1}{\beta^2}+N[d_1(2N+1)+|q|d(N+1)(2N+1)$$
$$+(|q|d+d_1)N^2]+\frac{M_1}{M}N(2N+1)d+(N-1)Nd[d|q|(N+1)+2d_1$$
$$+(|q|d+d_1)(2N-1)]+\frac{M_1}{M}(N-1)Nd^2+(N-2)(N-1)N(|q|d+d_1)d^2\Big\}$$
$$+\frac{8}{3}\alpha M^2(N+1)(1+2d+2d^2) \le \frac{4}{3}M(N+1)\Big[\frac{|q|d}{\beta^2}+\frac{d_1}{\beta^2}+3d_1+4|q|d$$
$$\frac{2M_1}{M}d+\frac{M_1}{M}d^2+3d^2|q|+4dd_1+2\alpha M(1+2d+2d^2)\Big].$$

Now, we estimate $|u_{N+1}|$.

$$|u_{N+1}(t)| \le e^{-b_{N+1}t}|u_{N+1,0}|+e^{-b_{N+1}t}\int_0^t |R_{N+1}|e^{b_{N+1}}\tau d\tau \le |u_{N+1,0}|$$
$$+\frac{4M}{3|q|}\Big[\frac{|q|d}{\beta^2}+\frac{d_1}{\beta^2}+3d_1+4|q|d+\frac{2M_1d}{M}+\frac{M_1d^2}{M}+3d^2|q|+4dd_1+2\alpha M(1+2d+2d^2)\Big]$$
$$\le M\Big[\frac{|u_{N+1,0}|}{M}+\frac{3}{10}+4d^2+\frac{17}{3}d+\frac{4}{15}d+8d^2+16d+\frac{1}{6}d+\frac{1}{6}d^2\Big]$$
$$\le M\Big(\frac{1}{2}+13d^2+23d\Big) \le M.$$

The last inequality is valid, since for $d = 0.02$ we have $13d^2+23d \le 0.5$.

Thus, the inequality (12) is proved. Now, we estimate $|u'_{N+1}|$.

$$|u'_i(t)| \leq |b_{N+1}| + |R_{N+1}| \leq M_1(N+1)\Big\{\frac{M}{M_1}(|q| + \alpha\alpha_0) + \frac{4}{3}\Big[2d + d^2$$

$$+\frac{M}{M_1}\Big(\frac{|q|d}{\beta^2} + \frac{d_1}{\beta^2} + 3d_1 + 4|q|d + 3d^2|q| + 4dd_1 + 2\alpha M(1 + 2d + 2d^2)\Big)\Big]\Big\}$$

$$\leq M_1(N+1)\Big\{\frac{4}{3}(2d+d^2) + \frac{M|q|}{M_1}\Big[1 + \frac{17}{4}d + 3d^2 + \frac{1}{24} + \frac{1}{5} + \frac{1}{3}(1+2d+2d^2)\Big]\Big\}$$

$$\leq M_1(N+1)\Big[\frac{2M|q|}{M_1} + \frac{4}{3}(2d+d^2)\Big] \leq M_1(N+1)\Big(\frac{2}{3} + \frac{1}{3}\Big) \leq M_1(N+1).$$

Now, we can prove the convergence of the series (6) to the solution of the Cauchy problem (8), (10) in the domain G_T, in which $|P(x,t)| < 1$. Theorem is proved.

Assertion 2. *Suppose that the conditions of the Theorem are satisfied and the initial condition* (10) *satisfies the inequality*

$$0 \leq u_0(x) \leq M + \sum_{i=1}^{\infty} u_{io} e^{-\varphi_0 i}, \quad (u_0 = M, \quad \varphi_0 > T|q| + 1).$$

Then in domain G_T the following inequality is valid:

$$0 \leq u(x,t) \leq M + \frac{M}{e-1} e^{-\frac{8}{3}\alpha M t}. \tag{14}$$

We can prove assertion 2 similar to the proof of the theorem, if we take inequalities

$$|u_i| \leq e^{-\frac{8}{3}\alpha M t} M, \quad |u'_i| \leq i e^{-\frac{8}{3}\alpha M t} M_1, \quad i \geq 1$$

instead of the inequalities (12), (13).

Remark 1. In paper [25] for initial-boundary problem (9) for equation (7) in domain Q_T={$(x,t) : 0 \leq x \leq L, \quad 0 \leq t \leq T$ by conditions

$$0 \leq u_0(x) \leq D_1, \quad D_1 > 0,$$

was proved following inequality

$$0 \leq u(x,t) \leq D_1 + D_2 t, \quad D_2 > 0. \tag{15}$$

Remark 2. Assertion 2 allows us to make the inequality (15) stronger, if we assume that special boundary conditions are given for $x = 0$ and $x = L$ generated by an arbitrary function $f(t)$. In this case we have the inequality (14).

CONCLUSION

A general scheme for constructing special series for representing solutions of a wide class of nonlinear partial differential equations is proposed. A new class of solutions of generalized Boussinesque equation with an arbitrary function in the form of a convergent series is constructed. It is shown that under special boundary conditions we can construct a solution of an initial-boundary problem with exponential asymptotic.

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